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Embedding optimal selection problems in a Poisson process

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We consider optimal selection problems, where the number N_1 of candidates for the job is random, and the times of arrival of the candidates are uniformly distributed in [0, 1]. Such best choice problems are generally harder than the fixed-N counterparts, because there is a learning process going on as one observes the times of arrivals, giving information about the likely values of N_1. In certain special cases, notably when N_1 is geometrically distributed, it had been proved earlier that the optimal policy was of a very simple form; this paper will explain why these cases are so simple by embedding the process in a planar Poisson process from which all the requisite distributional results can be read off by inspection. Routine stochastic calculus methods are then used to prove the conjectured optimal policy.

record processes * optimal selection * martingales * stochastic calculus * planar Poisson process

1. Introduction

Our main attention here will be on certain variants of the classical secretary problem previously studied by Gianini and Samuels (1976), Cowan and Zabczyk (1978), Stewart (1981), and in more generality, by Bruss (1984) and Bruss and Samuels (1987). In the latter examples N secretaries present themselves for a job interview, at times $T_1, T_2, \ldots, T_N$. The number N will in general be a random variable, but conditioned on N we assume that the T_j are independent, uniformly distributed on $[0, 1]$. We suppose that secretary 1 is better than secretary 2 is better than ... is better than secretary N, and that the selector can rank any subset of the secretaries by merit, but cannot tell how the candidates of this subset rank relative to the whole sample of N.

In the simplest version of the problem the selector may choose no more than one candidate and is penalised according to the rank of the chosen candidate among the whole sample; a loss of 0 for choosing the best and 1 otherwise, is commonly used (and called the best-choice problem).

A candidate arriving at time t who is the kth best among those candidates arriving at or before time t will be referred to as a *k-record*. We let μ_k denote the point process (=random measure) of k-records, so that

$$\mu_k((0, t]) \equiv \text{no. of } k\text{-records arriving in } (0, t].$$

The theory of record processes proves a useful tool in solving optimal selection problems based on relative ranks; an example in Gaver (1976), and Bruss (1988) develops the ideas further in a variety of examples.

We shall denote the loss incurred by accepting a k-record which arrives at time t by $q_k(t)$, and by a *choice process H* we shall understand a sequence of $\{0, 1\}$-valued previsible processes $(H_k(t))_{t\geq 0}$, $k = 1, 2, \ldots$. The interpretation is that $H_k(t) = 1$ if and only if the selector is prepared to accept a k-record which arrives at time t. The choice process H must be previsible with respect to the filtration $(F_t)_{t\geq 0}$ generated by the k-record processes; see, for example, Brémaud (1981) for background on the general theory of processes, with particular relevance to point processes. The previsibility restriction on H is technically convenient, and conceptually inoffensive; if one were going to accept a k-record at time t, one should be able to say so just before time t!

With these conventions, then, the overall loss associated to a choice process H is

$$C = \sum_{k\geq 1} \int_0^1 H_k(t) q_k(t) \mu_k(\mathrm{d}t) + Z,$$

where Z is some termination loss. Only choice processes H satisfying the obvious condition for one choice

$$H_k(t) = 0 \quad \text{for all } t > \tau \equiv \inf\left\{ u\colon \sum_{k\geq 1} \int_0^u H_k(s)\mu_k(\mathrm{d}s) = 1 \right\}$$

are admissible. The aim is to choose an admissible choice process to minimise $E(C)$.

Notice immediately that the processes q_k are *not adapted* to the filtration $(F_t)_{t\geq 0}$; only at time 1 can one know for sure what the loss associated with choosing a k-record at time t will be.

But, because H and μ are optional processes,

$$E(C) = E\left[\sum_{k\geq 1} \int_0^1 H_k(t)\mu_k(\mathrm{d}t)\bar{q}_k(t) + Z \right],$$

where $\bar{q}_k(t) = E[q_k(t) | F_t]$ is the optional projection of q_k.
Thus it is equivalent and more convenient to minimise $E(\bar{C})$, where

$$\bar{C} \equiv \sum_{k\geq 1} \int_0^1 H_k(t)\bar{q}_k(t)\mu_k(\mathrm{d}t) + Z.$$

In order to solve this optimisation problem in any explicit way, then, it is going to be essential to know about

the distribution of the k-record processes μ_k; (1a)

the form of $q_k(\cdot)$. (1b)

The main contribution of this paper is to show that by embedding suitably in a Poisson process we can, in a number of cases, quickly and easily obtain this information. A particular advantage of this approach is that the limiting results as $N \to \infty$ can be read off immediately, because they are actually almost sure limits and not just distributional limits. Indeed the case $N = \infty$ turns out to be technically and conceptually the simplest, and is the one we treat first, in Section 2. Our approach allows us to prove an 'infinite secretary problem' of Gianini and Samuels (1976) by verifying their solution.

In Section 3, we turn to the case where N has a geometric distribution, a case already identified by Bruss and Samuels (1987) as being special in the sense that the risk of accepting an arrival only depends on its relative rank among those arrived so far and on the time, but not on the history of the arrival process. We shall see that this is explicable by the nice way in which this problem is embedded in the Poisson point process, which leads us to their solution of the optimal selection problem.

Section 4 studies the case of an arbitrary distribution of N. The embedding in a Poisson process now helps us little, but the power and the flexibility of the general theory formulation which we have adopted takes over and allows us to extend the earlier results, and obtain the characterisation of an optimal policy given by Bruss and Samuels (1990). In view of the complete generality of the law of N, it is not surprising that the solution is quite complicated, and the proof we give is of the familiar but non-informative variety "Consider the function f defined by . . .". This is typical in optimal control problems where frequently one has a candidate for the optimal control and the value function obtained by some simple argument, and then confirms that these candidates are correct by checking that they give a submartingale in general, and a martingale under supposed optimal control. Such is the situation here, where Bruss and Samuels (1990) have obtained (by an approximate Bellman equation) the form of the solution. What Section 4 adds however is the conclusion that routine stochastic calculus provides a firm framework within which to prove such results.

2. Poisson embedding and the infinite-secretary-problem

Let $Y \equiv (-\infty, 0)$, and let ν be a Poisson random measure in $(0, \infty) \times Y$ with expectation measure equal to Lebesgue measure. If ν puts a unit mass on (t, y) we interpret this as the arrival at time t of a candidate of *quality* y. This Poisson random measure (or equivalently, Poisson point process) is the natural setting for the study of the optimal selection problems described above, with a random number of arrivals coming at independent U[0, 1]-times; for example, if we wanted the total number N of arrivals to have a Poisson(λ) distribution, we would simply ignore all points of ν with qualities below $-\lambda$. But for now we do not do this, concentrating instead on the whole of the Poisson point process ν, to obtain the following result.

Theorem 1. *The k-record processes* $(\mu_k)_{k\geq 1}$ *are independent inhomogeneous Poisson processes on* $(0, \infty)$ *with expectation measure* $t^{-1}\,dt$. *The distribution of the number of points in the time interval* $(t, 1)$ *with quality higher than that of the kth best point in the time interval* $(0, t)$ *is negative binomial given by*

$$P(\text{there are } j \text{ points in } (t,1) \text{ better than the } k\text{th best in } (0,t))$$
$$=\binom{j+k-1}{k-1}t^k(1-t)^j. \tag{2}$$

Proof. We prove the first statement first. Recall that a point is a k-record if and only if there are exactly $(k-1)$ better points in the point process which arrive earlier. We are using μ_k to denote the point process of times at which a k-record occurs, and ν_k to denote the point process on $(-\infty, 0)$ of the *qualities* of k-records. (Figure 1 illustrates the situation for $k=1$).

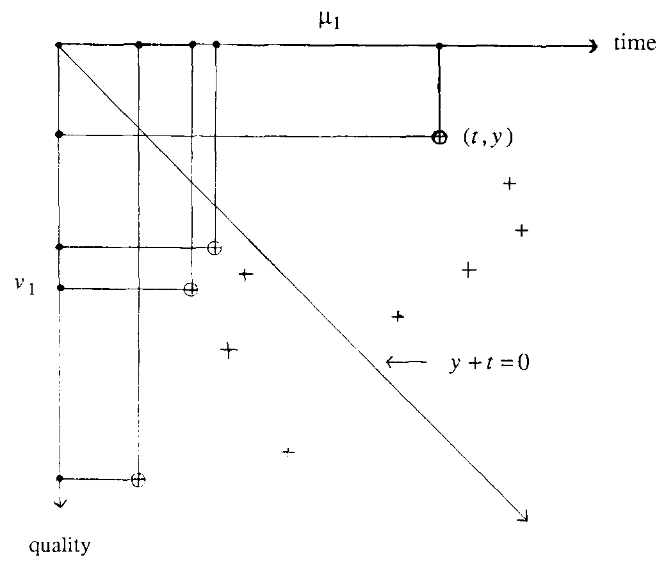

Fig. 1.

Now it is evident that if we reflect ν in the line $y+t=0$ (so in other words, we transform (t, y) to $(-y, -t)$), to obtain a point process ν^*, then ν and ν^* have the same law. Moreover any point of ν which is a k-record is mapped to a k-record for ν^*. But this transformation carries the point process μ_k of k-record *times* for ν to the point process ν_k^* of k-record *qualities*. Thus the law of the μ_k is exactly the same as the law of the $\tilde{\nu}_k$, where $\tilde{\nu}_k$ is the reflection of ν_k via $(t, y)\to(-t, -y)$ into $(0, \infty)$. But the law of the ν_k is easy to obtain from Ignatov's Theorem: by Corollary 5.1 of Goldie and Rogers (1984) they are independent inhomogeneous Poisson processes with expectation measure $|t|^{-1}\,dt$, from which the first statement follows immediately.

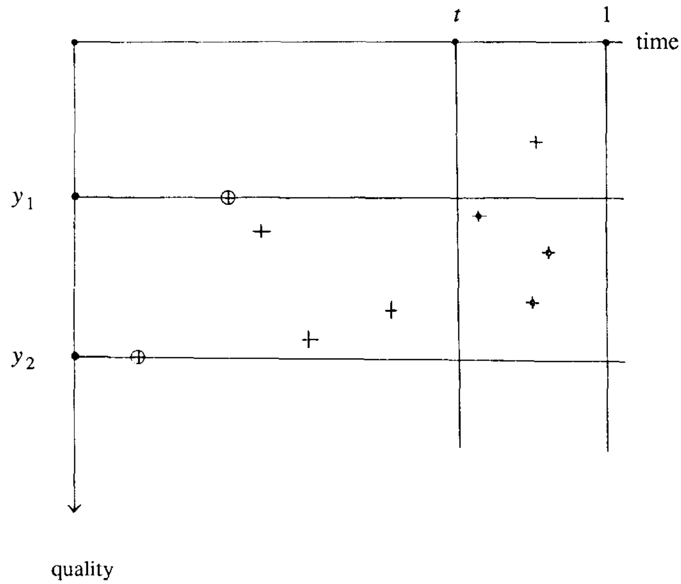

Fig. 2.

The second statement is easy to deduce from an inspection of Figure 2.

Let us fix $t \in (0, 1)$ and consider the qualities $Y_1 > Y_2 > \cdots$ of the points which occur in the time interval $(0, t)$. By a well-known property of Poisson processes, the spacings $Y_j - Y_{j+1}$ are independent exponentials with parameter t; thus the number V_j of points arriving between the times t and 1 with quality between Y_j and Y_{j+1} will be geometric with parameter t, because

$$\begin{aligned} P(V_j = r) &= \int_0^\infty t\,\mathrm{e}^{-ty}\,\mathrm{d}y\,P(V_j = r \mid Y_j - Y_{j+1} = y) \\ &= \int_0^\infty t\,\mathrm{e}^{-ty}\,\mathrm{d}y\,((1-t)y)^r\,\mathrm{e}^{-(1-t)y}/r! = t(1-t)^r. \end{aligned}$$

Equation (2) now follows immediately. □

Theorem 1 allows us to prove the result of Gianini and Samuels (1976) on optimal stopping of an "infinite secretary problem", where a loss $q(j)$ is incurred if the candidate chosen is the (definite) rank j, and a loss $q(\infty) \equiv \sup\{q(j) \mid j \geq 1\}$, if no candidate is selected. Only one candidate may be chosen. It is reasonable to suppose that q is non-decreasing. We also suppose that $q(\infty)$ is finite.

By the second part of Theorem 1, recalling that $q_k(t)$ is the actual loss incurred if at time t we accept the kth best to date, we have

$$\begin{aligned} \bar{q}_k(t) &\equiv E(q_k(t) \mid F_t) \\ &= \sum_{j \geq 0} q(k+j) P(j \text{ points in } (0, t) \text{ better than the } k\text{th best in } (0, t)) \end{aligned}$$

$$= \sum_{j \geq 0} q(k+j) \binom{j+k-1}{k-1} t^k (1-t)^j,$$

where the last equality follows from (2).

Now let f solve

$$f'(t) = t^{-1} \sum_{j=1}^{\infty} [f(t) - \bar{q}_j(t)]^+, \quad \text{if } 0 < t \leq 1, \qquad f(1) = q(\infty). \tag{3}$$

According to Gianini and Samuels (1976, Proposition 5.4) there is a unique solution to this differential equation.

Theorem 2. (i) *An optimal choice process for the infinite secretary problem is given by*

$$H_k^*(t) = \begin{cases} 0, & \text{if } t > \tau \text{ or if } \bar{q}_k(t) > f(t), \\ 1, & \text{if } t \leq \tau \text{ and } \bar{q}_k(t) \leq f(t), \end{cases}$$

where $\tau = \inf\{u : \sum_k \int_0^u H_k(s)\ \mu_k(\mathrm{d}s) > 0\}$ *is the time at which the choice is made.*

(ii) *The minimal expected loss is* $f(0)$.

Proof. Let $\tau_t = \min\{\tau, t\}$. Then the process

$$V_t \equiv f(t) I_{\{t<\tau\}} + \sum_k \int_{(0,\tau_t]} H_k(s) \bar{q}_k(s) \mu_k(\mathrm{d}s) + q(\infty) I_{\{t=\tau=1\}} \tag{4}$$

is a martingale when $H = H^*$, and is a submartingale in general. To see this we firstly note that if H is a permissible choice process, then each term in the expression for V is bounded by $q(\infty)$. Thus V is well-defined. Secondly, since $\mu_k(\mathrm{d}t)$ is compensated by $t^{-1}\,\mathrm{d}t$, we may, by adding a martingale, change to

$$U_t \equiv f(t) I_{\{t<\tau\}} + \sum_k \int_{(0,\tau_t]} H_k(s) \bar{q}_k(s) s^{-1}\,\mathrm{d}s + q(\infty) I_{\{t=\tau=1\}}. \tag{5}$$

(In general, $\bar{q}_k$ is only an optional process, but in this particular case it is deterministic, therefore previsible.) The task is thus to show, that U is a submartingale, and a martingale if $H = H^*$.

Now the finite variation process $I_{\{t<\tau\}}$ is compensated by

$$-\sum_k \int_0^{\tau_t} H_k(s) s^{-1}\,\mathrm{d}s.$$

Thus, expanding $\mathrm{d}U$ by Ito's formula, we obtain

$$\begin{aligned} \mathrm{d}U_t &= \left[f'(t) + \sum_k H_k(t) \bar{q}_k(t) t^{-1} - f(t) \sum_k H_k(t) t^{-1} \right] I_{\{t<\tau\}}\,\mathrm{d}t \\ &\quad + \mathrm{d}(\text{martingale}) \\ &= t^{-1}\,\mathrm{d}t\, I_{\{t<\tau\}} \sum_k [[f(t) - \bar{q}_k(t)]^+ - [f(t) - \bar{q}_k(t)] H_k(t)] \\ &\quad + \mathrm{d}(\text{martingale}). \end{aligned}$$

The first term is the differential of an increasing process, so that U is a submartingale. Moreover, if $H_k = H_k^*$, then the first term vanishes and U is a martingale, as required. And finally, the last term in (5) is obviously a submartingale, which is a martingale if $H_k = H_k^*$. This completes the proof. □

As an example, if $q(j) = 1$ for $j > 1$ and $q(1) = 0$, we have the familiar best choice problem. Here $q_1(t) = 1 - t$ and $q_j(t) = 1$ for $j > 1$. The differential equation (3) is solved by $f(t) = 1 + t \ln t$ for $t \geq 1/\mathrm{e}$ and $f(t) = 1 - 1/\mathrm{e}$ for $t \leq 1/\mathrm{e}$ so that the optimal loss (i.e. the value of the game) equals $1 - 1/\mathrm{e}$ and the optimal cutoff time $t^* = 1/\mathrm{e}$. (Compare with Gianini and Samuels (1976).) We also note that this is the worst case in the sense that according to the $1/\mathrm{e}$-law (Bruss, 1984) this optimal loss is an upper bound for the optimal loss for arbitrary distribution of N.

3. Geometric distribution of N

As we described at the beginning of Section 2, we could model the process of arrivals and their qualities for finite random N simply by taking some random variable Z and looking only at arrivals whose quality exceeds $-Z$. This is essentially the censoring argument of Bruss and Samuels (1987, p. 825), but in this section we present a way to look at censoring which makes it perfectly clear why certain cases become 'nice', as it is the case of the geometric distribution. If we take Z to have an exponential distribution, independent of the Poisson random measure ν, then the distribution of N is geometric, since

$$P(N = k) = \int_0^\infty a\,\mathrm{e}^{-az}\,\mathrm{d}z\, z^k\,\mathrm{e}^{-z}/k! = a(1+a)^{-k},$$

where a is the parameter of the exponential $(E(Z) = 1/a)$.

However, we can get the distributions of the k-record process, and of the number of arrivals better than a k-record just by redrawing the picture of the Poisson random measure ν. The essential observation is that the negative of the quality of the best candidate to arrive by time $a > 0$ has an exponential distribution with parameter a, so we could use the best candidate before time a to provide our exponentially distributed cut-off, and start the time of the selection process at $a > 0$, running it until time $1 + a$. The process $\tilde{\mu}_1$ of 1-records which we now observe is just the process μ_1 of 1-records which occur in the time interval $(a, a+1)$; so for our selection process with a geometric random number of arrivals, the process of 1-records is an inhomogeneous Poisson process with expectation measure $(a+t)^{-1}\,\mathrm{d}t$.

What about the process of 2-records $\tilde{\mu}_2$? The situation is illustrated in Figure 3; it is slightly more complicated than the case of the 1-records. Every point of $\tilde{\mu}_2$ is going to be a point of μ_2, the 2-record process of ν, but there will be points of μ_2 which do not appear in $\tilde{\mu}_2$ because their quality is too low. A moment's thought shows, however, that a 2-record of ν is of *too low a quality* to appear in $\tilde{\mu}_2$ if and

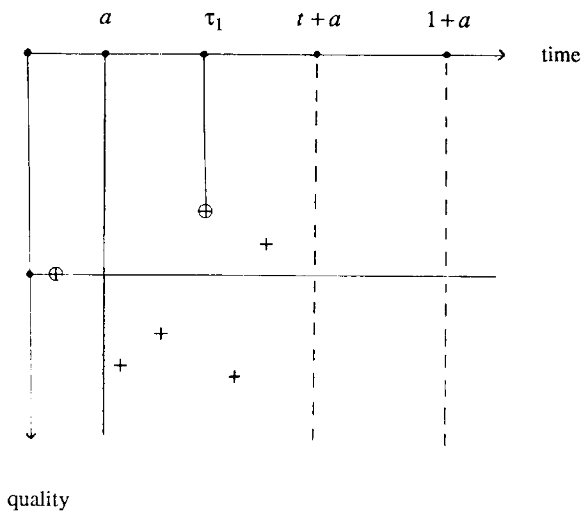

Fig. 3.

only if the corresponding 2-record time is *earlier* than τ_1, the first 1-record time after a. Hence if N_t denotes the number of candidates viewed by time t (which is equal to the number of points of ν arriving between a and $a+t$ with quality better than the best before time a), we have informally that

$$\tilde{\mu}_2 \text{ is inhomogeneous Poisson of rate } I_{\{N_t>0\}}(a+t)^{-1}\,\mathrm{d}t.$$

We can make this precise by defining

$$\mu'_k((0,t]) \equiv \mu_k((a, a+t]),$$

so that $(\mu'_k)_{k\geq 1}$ are independent inhomogeneous Poisson processes with rate $(a+t)^{-1}\,\mathrm{d}t$, then setting

$$T_0 = 0, \qquad T_{k+1} \equiv \inf\left\{t>0\colon \sum_{j=1}^{k+1} \mu'_j((T_k, t]) > 0\right\}, \quad k \geq 0,$$

and finally defining

$$\tilde{\mu}_k(\mathrm{d}t) \equiv I_{\{t>T_{k-1}\}}\mu'_k(\mathrm{d}t)$$

Recall that we want to compute $\bar{q}_k(t)$, for example. Using simple properties of the Poisson process, we shall prove that

$$P(\text{exactly } j \text{ observations between } t \text{ and } 1 \text{ exceed } k\text{th best in } (0,t))$$
$$= \binom{j+k-1}{k-1}\left(\frac{a+t}{a+1}\right)^k \left(\frac{1-t}{a+1}\right)^j.$$

By letting $a \downarrow 0$, we recover (2), as of course we must. To prove this formula, let $A_1 = (0, a]$, $B = (a, a+t]$, $A_2 = (a+t, a+1]$ and consider the original Poisson random measure. Let $A = A_1 \cup A_2$. By the argument which gave (2),

$$P(F) = \left(\frac{t}{a+1}\right)^k \left(\frac{a+1-t}{a+1}\right)^j \binom{j+k-1}{k-1},$$

where F is the event {exactly j candidates in A are better than the kth best in B}. Next, if G is the event {no candidate in A_1 is better than the kth best in B}, then

$$P(G \mid F) = P(\text{best } j \text{ in } A \text{ all fall in } A_2) = \left(\frac{1-t}{a+1-t}\right)^j.$$

Lastly,

$$P(k\text{th-best in } B \text{ is better than all in } A_1) = \left(\frac{t}{a+t}\right)^k.$$

Hence the conditional probability of the event that exactly j in A_2 are better than the kth-best in B given that the kth best in B is better than all in A_1 is given by

$$\binom{j+k-1}{k-1}\left(\frac{t}{a+1}\right)^k \left(\frac{a+1-t}{a+1}\right)^j \left(\frac{1-t}{a+1-t}\right)^j \left(\frac{t}{a+t}\right)^{-k},$$

which reduces to the stated expression.

4. The general problem

Let us now return to the general problem, where there is a finite random number N of candidates, whose arrival times are independent and uniformly distributed through $(0, 1)$.

Let N_t denote the number of candidates that arrive at or before time t, so that $N_0 = 0$ and $N_1 = N$. Suppose for the moment that N was *known*. Then

$$M_t' = N_t - \int_0^t (N - N_s)(1-s)^{-1}\,\mathrm{d}s$$

is a martingale relative to the filtration $G_t = (F_t) \wedge (\sigma(N))$. However the selector only has the information in F_t, which at time t tells him the times of arrival and relative ranks of candidates so far. Thus if we take the F_t-optional projection of M', we find that

$$M_t = N_t - \int_0^t (E(N \mid F_s) - N_s)(1-s)^{-1}\,\mathrm{d}s \text{ is a martingale.}$$

If the distribution of N is given by $P(N = n) = p_n$ then elementary calculations give

$$P(N - N_s = j \mid F_s) = p_{n+j}\binom{n+j}{j} s^n (1-s)^j c_n(s),$$

where we have abbreviated N_s to n, and where $c_n(s)$ is the appropriate normalising constant. Hence we can write down an expression for

$$E(N-N_s\,|\,F_s)=\sum_{j\geq 0} jp_{n+j}\binom{n+j}{j}s^n(1-s)^j c_n(s)\equiv\psi(s, N_s),$$

say. (We shall assume always that ψ is finite-valued. Note that if the distribution of N_1 is geometric, $p_n=a(1+a)^{-n-1}$, then $\psi(t, n)=n(1-t)/(t+a)$). Likewise it is elementary, that if $N_t=k$, then the relative rank of the next candidate to arrive is equally likely to be any of $1, 2, \ldots, k+1$. Hence for each k,

$$M_t^k\equiv\mu_k((0, t])-\int_0^t (1+N_s)^{-1}\psi(s, N_s)I_{\{N_s\geq k-1\}}(1-s)^{-1}\,\mathrm{d}s \tag{6}$$

is a martingale.

Recall that we envisage a situation where only one candidate may be chosen, and that if the definite rank j is chosen, a loss $q(j)$ is incurred, where q is non-decreasing. Let us suppose that we incur the loss $Q(N)$ if we fail to choose any of the candidates. Recall also that in the optimisation problem we needed to make explicit $\bar{q}_k(t)\equiv E(q_k(t)\,|\,F_t)$; in this example, elementary arguments show that $\bar{q}_k(t)=\rho_k(t, N_t)$ for some deterministic functions ρ_k, which can be written down in terms of p_n and t, though it profits us little in general.

Bruss and Samuels (1990) express the optimal solution to the problem in terms of the solution of the system of differential equations

$$f_k'(t)=(1-t)^{-1}\psi(t, k)\left[f_k(t)-\frac{1}{k+1}\sum_{j=1}^{k+1}\min\{f_{k+1}(t), \rho_j(t, k+1)\}\right] \tag{7}$$

and

$$f_k(1)=Q(k), \quad k\geq 0.$$

(In fact, they do not discuss questions of existence and uniqueness of solutions, which are not relevant in their treatment, but it is easy to prove, that if q is bounded (as we shall suppose), then (7) has a unique solution. To see this, one constructs an nth approximation to the solution by supposing that $f_k(\cdot)\equiv\infty$ for all $k>n$, and observes that for each k, the approximations, $f_k^{(n)}$, say, decrease with n. We leave details to the reader.)

As before, we let

$$\tau\equiv\min\left\{1, \inf\left\{t\colon \sum_{k\geq 1}\int_{(0, t]} H_k(s)\mu_k(\mathrm{d}s)>0\right\}\right\}$$

be the time when the decision process stops. According to Bruss and Samuels, an optimal choice process is given by

$$H_k^*(t)=\begin{cases}1, & \text{if } \rho_k(t, N_t)\leq f_{N_t+1}(t) \text{ and } t\leq\tau,\\ 0, & \text{otherwise.}\end{cases}$$

We confirm this by proving that for any permissible choice process H,

$$V_t \equiv \sum_{k\geq 1} \int_0^{\min\{t,\tau\}} H_k(s)\,\rho_k(s, N_s)\mu_k(\mathrm{d}s) + I_{\{t<\tau\}} \sum_{n\geq 0} f_n(t) I_{\{N_t=n\}} + Q(N_1) I_{\{t=\tau=1\}} \tag{8}$$

is a submartingale, and a martingale if $H = H^*$. Notice that the process $\rho_k(t, N_t)$ is in general only optional, but we may replace it by the previsible process $\rho_k(t, N_{t-}+1)$ without changing (8). We shall do this.

By definition of f_n, there is no discontinuity of V at $t=1$, so we are only going to be concerned with jumps of V in $(0, 1)$. If we let $T_1 < T_2 < \cdots$ be the successive arrival times of the candidates, we have

$$V_t = \sum_k \int_0^{\min\{t,\tau\}} H_k(s)\,\rho_k(s, N_{s-}+1)\mu_k(\mathrm{d}s) + \sum_n f_n(t) I_{\{T_n \leq t < \min\{T_{n+1},\tau\}\}} + Q(N_1) I_{\{t=\tau=1\}}$$

or, in view of (6),

$$V_t \underset{\text{mar}}{=} \sum_k \int_0^{\min\{t,\tau\}} H_k(s)\,\rho_k(s, N_s+1)(1+N_s)^{-1}\psi(s, N_s) I_{\{N_s\geq k-1\}}(1-s)^{-1}\,\mathrm{d}s + \sum_n f_n(t) I_{\{T_n\leq t<\min\{T_{n+1},\tau\}\}} + Q(N_1) I_{\{t=\tau=1\}}, \tag{9}$$

where $\underset{\text{mar}}{=}$ signifies, that the two sides differ by a martingale. Now

$$\begin{aligned}
&\mathrm{d}(f_n(t) I_{\{T_n\leq t<\min\{T_{n+1},\tau\}\}}) \\
&\quad \underset{\text{mar}}{=} f_n'(t) I_{\{T_n\leq t<\min\{T_{n+1},\tau\}\}}\,\mathrm{d}t \\
&\qquad + f_n(t) I_{\{N_t=n-1\}}\psi(t, N_t)(\mathrm{d}t/(1-t)) I_{\{t<\tau\}} \\
&\qquad \times \Big(1 - \sum_k H_k(t)(1+N_t)^{-1} I_{\{N_t\geq k-1\}}\Big) \\
&\qquad - f_n(t) I_{\{N_t=n\}}\psi(t, N_t)(\mathrm{d}t/(1-t)) I_{\{t<\tau\}}.
\end{aligned}$$

When we sum over $n \geq 0$ we get to within martingales

$$\begin{aligned}
I_{\{t<\tau\}}\,\mathrm{d}t \Bigg[& f'_{N(t)}(t) + f_{N(t)+1}(t)\psi(t, N_t)(1-t)^{-1}\Big(1 - \sum_{k=1}^{N(t)+1} H_k(t)(1+N_t)^{-1}\Big) \\
& - f_{N(t)}\psi(t, N_t)(1-t)^{-1}\Bigg],
\end{aligned}$$

and hence, using the relation (7) for f_n',

$$\begin{aligned}
\mathrm{d}V_t \underset{\text{mar}}{=} & I_{\{t<T\}}\psi(t, N_t)(1-t)^{-1}\,\mathrm{d}t \\
& \times\Bigg[-(1+N_t)^{-1}\sum_{k=1}^{N(t)+1}\min\{f_{N(t)+1}(t), \rho_k(t, N_t+1)\} \\
& \quad +f_{N(t)+1}(t)\Bigg(1-\sum_{k=1}^{N(t)+1} H_k(t)(1+N_t)^{-1}\Bigg) \\
& \quad +\sum_{k=1}^{N(t)+1} H_k(t)\rho_k(t, N_t+1)(1+N_t)^{-1}\Bigg] \\
= & I_{\{t<T\}}\psi(t, N_t)(1-t)^{-1}(1+N_t)^{-1}\,\mathrm{d}t \\
& \times\sum_{k=1}^{N(t)+1}\{H_k(t)[\rho_k(t, N_t+1)-f_{N(t)+1}(t)]+[f_{N(t)+1}(t)-\rho_k(t, N_t+1)]^+\},
\end{aligned}$$

which is evidently the differential of an increasing process, and is zero if $H=H^*$.

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